Applied Lattice Theory: Formal Concept Analysis

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The "Formal Concept Analysis" project was born around 1980, when a research group in Darmstadt, Germany, begun to systematically develop a framework for lattice theory applications. It was first presented to the mathematical public in a programmatic lecture given at the 1981 Banff conference on Ordered Sets [5]. Since then, several hundred articles [1] have been published, including a textbook on the mathematical foundations [2]. The Darmstadt group alone has participated in more than a hundred application cooperation projects. Former members of that team have founded a small firm and now make their living from such applications.

The sophisticated name of "Formal Concept Analysis" needs to be explained. The method is mainly used for the *analysis* of data, i.e. for investigating and processing explicitly given information. Such data will be structured into units which are formal abstractions of *concepts* of human thought allowing meaningful and comprehensible interpretation. We use the prefix *formal* to emphasize that these formal concepts are mathematical entities and must not be identified with concepts of the mind. The same prefix indicates that the basic data format, that of a *formal context,* is merely a formalization that encodes only a small portion of what is usually referred to as a "context".

Much of the mathematics required for the applications has been borrowed directly from lattice theory. The basic construction of a complete lattice from a binary relation is explained already in the first edition of Birkhoff's *Lattice Theory.* But the new goal made it also necessary to extend and to smoothen the theory. Thereby, Formal Concept Analysis has created results that may be of interest even without considering the applications by which they were motivated.

For proofs, citations, and further details we refer to [2].

1 Formal contexts and concept lattices

A triple (G, M, I) is called a **formal context** if G and M are sets and $I \subseteq G \times M$ is a binary relation between G and M. We call the elements of G **objects**, those of M **attributes**, and I the **incidence** of the context (G, M, I). For $A \subseteq G$, we define

$$A' := \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}$$

and dually, for $B \subseteq M$,

$$B' := \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}.$$

It is easy to prove that these **derivation operators** satisfy the following simple rules (for all $A_1, A_2, A \subseteq G$ and all $B_1, B_2, B \subseteq M$):

1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ 1') $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$

2) $A \subseteq A''$ and $A' = A'''$ 2') $B \subseteq B''$ and $B' = B'''$

$$A \subseteq B' \iff B \subseteq A'.$$

The experienced reader will have noticed that these derivation operators establish a Galois connection between the power set lattices on G and M and thereby a dual isomorphism between two closure systems. It is natural to consider the elements of this dual isomorphism, as is done in the following definition:

A pair (A, B) is a **formal concept** of (G, M, I) if and only if

$$A \subseteq G, B \subseteq M, A' = B, \text{ and } A = B'.$$

A is called the **extent** and B the **intent** of the concept (A, B). The concepts of a given context are naturally ordered by the **subconcept-superconcept relation** defined by

$$(A_1, B_1) \leq (A_2, B_2) :\iff A_1 \subseteq A_2 \quad (\iff B_2 \subseteq B_1).$$

The ordered set of all formal concepts of (G, M, I) is denoted by $\underline{\mathfrak{B}}(G, M, I)$ and is called the **concept lattice** of (G, M, I).

Concept lattices are indeed lattices. The following theorem shows that, more precisely, the concept lattices are, up to isomorphism, exactly the complete lattices. Every concept lattice is complete, and every complete lattice is isomorphic to some concept lattice. (Using topological contexts, non-complete lattices can be represented, too.)

Theorem 1 (The basic theorem on concept lattices) *The concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right)$$

$$\bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcup_{t \in T} A_t \right)'', \bigcap_{t \in T} B_t \right).$$

A complete lattice $\underline{L}$ is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\tilde{\gamma} : G \to L$ and $\tilde{\mu} : M \to L$ such that $\tilde{\gamma}(G)$ is supremum-dense in $\underline{L}$, $\tilde{\mu}(M)$ is infimum-dense in $\underline{L}$ and gIm is equivalent to $\tilde{\gamma}g \leq \tilde{\mu}m$ for all $g \in G$ and all $m \in M$. In particular, $\underline{L} \cong \underline{\mathfrak{B}}(L, L, \leq)$.

Concept lattices can be depicted by the usual lattice diagrams. It would however be too messy to label each concept by its extent and its intent. A much simpler **reduced labelling** is achieved if each object and each attribute is entered only once in the diagram. The name of object g is attached to the lower half of the corresponding **object concept** $\gamma g := (\{g\}'', \{g\}')$, while the name of attribute m is located at the upper half of the **attribute concept** $\mu m := (\{m\}', \{m\}'')$. It is then still possible to read off all extents and all intents from the diagram: for any concept (A, B), we have

$$g \in A \iff \gamma g \leq (A, B) \quad \text{and} \quad m \in B \iff (A, B) \leq \mu m.$$

In other words, extent and intent of an arbitrary concept can be found as the set of objects in the principal ideal and as the set of attributes in the principal filter generated by that concept.

Figure 1 shows an example of such a concept lattice diagram with reduced labelling. The formal context is not explicitly given but can easily be read off from the diagram. The objects here are seven finite lattices (given by their diagrams), the attributes are certain properties which a given lattice may or may not have. The incidence is as expected: a lattice is incident with a property if and only if it has it. It can be read from the diagram according to the general rule

$$(g, m) \in I \iff \gamma g \leq \mu m.$$

The **arrow relations** of a context (G, M, I) are defined as follows: for $h \in G$, $m \in M$, let

$$g \swarrow m \quad :\iff \quad \begin{cases} (g, m) \notin I \text{ and} \\ \text{if } g' \subseteq h' \text{ and } g' \neq h', \text{ then } hIm, \end{cases}$$

$$g \nearrow m \quad :\iff \quad \begin{cases} (g, m) \notin I \text{ and} \\ \text{if } m' \subseteq n' \text{ and } m' \neq n', \text{ then } gIn, \end{cases}$$

$$g \swarrow\!\!\!\nearrow m \quad :\iff \quad g \swarrow m \text{ and } g \nearrow m.$$

For given $g \in G$ there exists an attribute $m \in M$ with $g \swarrow m$ if and only if γg is $\bigvee$-irreducible. Dually, $g \nearrow m$ holds for some $g \in G$ if and only if μm is $\bigwedge$-irreducible. Therefore if we define a formal context to be **doubly founded** if

$$(g, m) \notin I \quad \Rightarrow \quad \exists_{n \in M}\, g \nearrow n \text{ and } m' \subseteq n'$$

and

$$(g, m) \notin I \quad \Rightarrow \quad \exists_{h \in G}\, h \swarrow m \text{ and } g' \subseteq h',$$

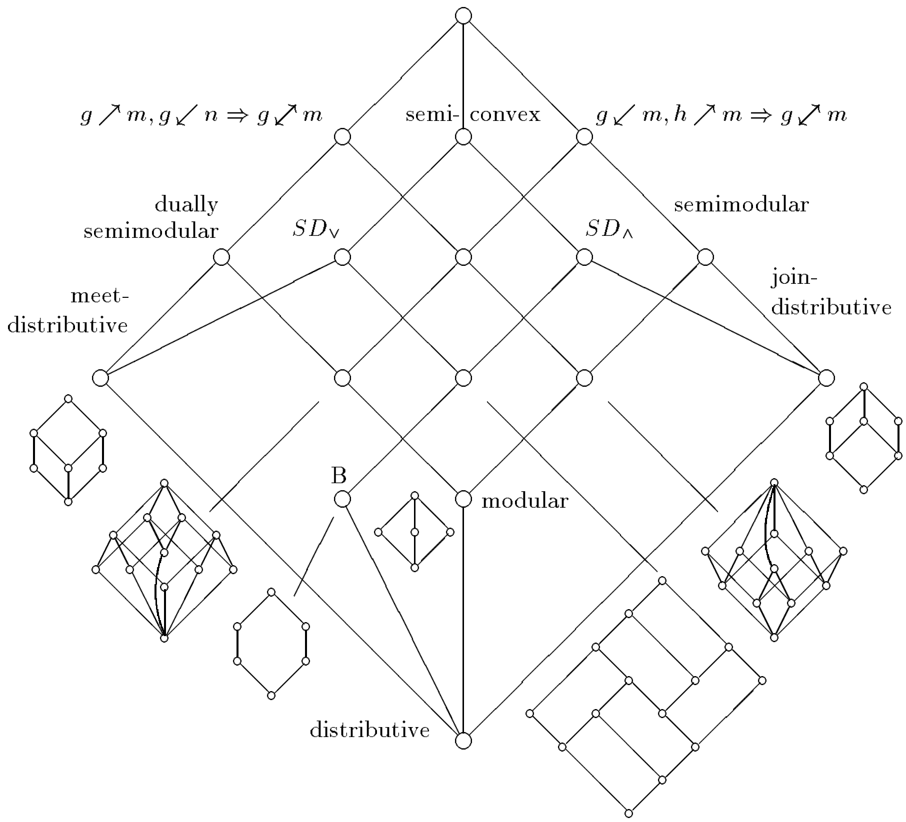


Figure 1: Generalizations of the distributive law. **B** abbreviates "bounded homomorphic image of a free lattice". A lattice is **semiconvex** if $x \wedge y = x \wedge z$ and $x \vee z = y \vee z$ together imply $x \leq z$. The arrow relations are defined in the text. The implications in this diagram hold for all finite lattices.

then the concept lattice of such a context will contain many irreducibles. It can in fact be shown that if $\underline{L}$ is such a lattice, then the set $J(\underline{L})$ of $\bigvee$-irreducibles is $\bigvee$-dense and the set $M(\underline{L})$ of $\bigwedge$-irreducibles is $\bigwedge$-dense, as in the case of finite lattices. By the basic theorem then $\underline{L}$ is isomorphic to the concept lattice of its **standard context** $(J(\underline{L}), M(\underline{L}), \leq)$. Every doubly founded context contains a subcontext isomorphic to the standard context; such a subcontext is called **reduced**. A sufficient condition for (G, M, I) to be doubly founded is that its concept lattice $\mathfrak{B}(G, M, I)$ is a **doubly founded lattice**, i.e. that for any two elements $x < y$ there are elements s and t such that s is minimal with respect to $s \leq y$, $s \nleq x$, and t is maximal with respect to $t \geq x$, $t \ngeq y$.

In order to construct all concepts of a finite formal context we recall that the extents form a closure system. The corresponding closure operator $A \mapsto A''$ can easily be computed from the formal context. Assuming $G := \{1, \ldots, n\}$, a linear strict order $<$ on the subsets of G is defined by

$$A < B :\iff A <_i B \quad \text{for some } i \in G,$$

where

$$A <_i B :\iff i \in B \setminus A \text{ and } A \cap \{1, \ldots, i-1\} = B \cap \{1, \ldots, i-1\}.$$

Moreover, let

$$A \oplus i := ((A \cap \{1, \ldots, i-1\}) \cup \{i\})''.$$

Theorem 2 *For $A \subseteq G$, $A \neq G$, the smallest closed set that is larger (with respect to $<$) than A, is $A \oplus i$, where i is the largest element of G satisfying $A <_i A \oplus i$.*

The theorem explains how to find the (lectically) next closed set, and thereby all closed sets: One starts with the closure $\emptyset''$ of the empty set and then repeatedly computes the next one, until the largest closed set, G, is reached. For a given formal context (G, M, I) this computes all extents, i.e. subsets $A \subseteq G$ satisfying $A = A''$, and thereby all concepts (A, A') of (G, M, I).

An **implication** $A \to B$ (between sets $A, B \subseteq M$ of attributes) **holds** in a formal context if and only if $B \subseteq A''$, i.e. if every object that has all attributes in A also has all attributes in B. The implications determine the concept lattice up to isomorphism and therefore offer an additional interpretation of the lattice structure. For practical purposes, it is desirable to reduce the set of all valid implications (which is usually huge) to a small set from which the others can be derived by inference. There is, in fact, a nice theory of **implication bases** for finite lattices. Particularly useful is the fact that such bases can be computed using the above algorithm.

2 Applications

It is our belief that the potential of Formal Concept Analysis as a branch of Applied Mathematics is just beginning to show. A typical task that concept lattices are useful for is to unfold given data, making their conceptual structure visible and accessible, in order to find patterns, regularities, exceptions, etc.

Most other techniques of data analysis have the aim to drastically reduce the given information and to obtain a few "significant parameters". By contrast, a concept lattice does not reduce complexity since it contains all the details of the data represented by the formal context. It is usually smaller than the power sets of G and M, since the two closure systems of extents and intents contain only those sets that are, in this specific sense, meaningful. Nevertheless it may be exponential in size, compared to the formal context. Complexity therefore is, of course, a problem, even though there are efficient algorithms and advanced program systems [4]. A formal context of moderate size may have more concepts than one would like to see individually. The mathematical theory therefore must provide tools to decompose, structure, browse through or navigate in a concept lattice. For reasons of efficiency it is desirable to develop methods that perform such lattice manipulations by working directly on the formal context.

The basic data type of a formal context is not the one occuring most frequently in the applications of Formal Concept Analysis. Often data is recorded in form of a tabular or relational data base, as described by the following definition:

A **many valued context** (G, M, W, I) consists of a set G (of "objects"), a set M (of "many valued attributes"), a set W (of "attribute values"), and a ternary relation $I \subseteq G \times M \times W$ satisfying

$$(g, m, v) \in I, (g, m, w) \in I \Rightarrow v = w.$$

A many valued context is unfolded into a lattice by first transforming it into a "one-valued" context, i.e. a formal context in the above sense, and then constructing the concept lattice of the latter. The transformation procedure is called **conceptual scaling**. It is not unique but contains several degrees of freedom that allow different interpretations. The simplest version, **plain scaling**, will be introduced here.

A **conceptual scale** for a many valued attribute m is a one-valued context which has the attribute values of m among its objects. With plain scaling, a scale is associated to each many valued attribute m, and m is replaced by the set of its scale attributes. Each value of m is substituted by the corresponding row of the scale. Formally, let (G, M, W, I) be a many valued context and, for each $m \in M$, let $\mathbb{S}_m := (G_m, M_m, I_m)$ be a scale for m. The **derived context** of (G, M, W, I) with respect to plain scaling with the scales $(S_m \mid m \in M)$ is given as (G, N, J), where

$$N := \bigtimes_{m \in M} \{m\} \times M_m$$

and

$$(g,(m,n)) \in J :\iff \exists_{w \in W}(g,m,w) \in I \text{ and } (w,n) \in I_m.$$

Nested line diagrams are an often useful variant of lattices diagrams. They are for example implemented in the commercial TOSCANA program system that uses concept lattices for browsing in data bases. As a small example, we show how such a diagram is obtained for the concept lattice of the formal context (G, M, I) given in figure 2.

At first, the attribute set M is split into sets $M_1 := \{a, b, c, d\}$, $M_2 := \{e, f, g, h, i\}$. Then the concept lattices $\underline{\mathfrak{B}}(G, M_1, I \cap G \times M_1)$ (which is an eight-element boolean lattice) and $\underline{\mathfrak{B}}(G, M_2, I \cap G \times M_2)$ (which is a cardinal sum of a three element chain with two singletons, plus 0 and 1) are computed. The diagram shown in figure 2 is that of a direct product of these two lattices, in which the elements of $\underline{\mathfrak{B}}(G, M, I)$ (which, by a general fact, is isomorphic to a $\bigvee$-subsemilattice of this product) are set off through the filled circles.

A dominant type of applications uses **conceptual data systems**. These consist of a many valued context stored in a (relational) data base, together with a collection of conceptual scales with line diagrams of their concept lattices. The conceptual knowledge encoded in a conceptual data system is expressed by its concept lattice; however, this lattice is in most cases far too large to be represented graphically. It therefore must be decomposed and displayed in small "aspects", the simplest of which are the concept lattices of the given conceptual scales. Suitable collections of these can be combined and displayed as nested line diagrams. This is performed automatically by the TOSCANA program system mentioned above: the user can choose a combination of conceptual scales, and a nested line diagram is generated and displayed. It is possible to "roam" through the data system by freely selecting scale combinations.

The many tasks that can be performed with this approach, like *exploring, searching, recognizing, identifying* etc., have been discussed in detail elsewhere [7]. Selected projects in different disciplines, most of which have been worked out in cooperation with the Darmstadt group, may eludicate the multitude of possible applications: *medicine:* diabetes of children, *psychosomatic:* repertory grids of anorexic patients, *psychology:* children's concept development, *musicology:* audio-visual perception of music, *politics:* international cooperations, *linguistics:* semantic structures of lexical data bases, *information science:* retrieval system for a library, *computer science:* software reengineering, *electronics:* improving chip production, *civil engineering:* retrieval system concerning laws and regulations, *ecology:* water pollution, *biology:* color perception. See [6] for references.

3 Sublattices and factor lattices

There are two natural ways to define substructures of a formal context (G, M, I). One is to omit some objects and attributes and to restrict the incidence to the

	a	b	c	d	e	f	g	h	i
1	×	×					×		
2	×	×					×	×	
3	×	×	×				×	×	
4	×		×				×	×	×
5	×	×		×		×			
6	×	×	×	×		×			
7	×		×	×	×				
8	×		×	×		×			

Figure 2: To obtain a nested line diagram, the attribute set is divided.

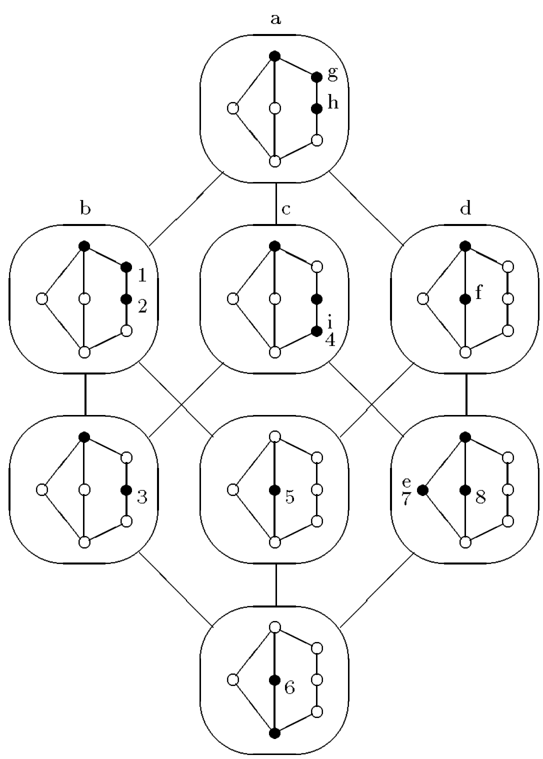


Figure 3: The filled circles represent the concept lattice of the formal context in figure 2.

remaining ones, i.e. to form a **subcontext** $(H, N, I \cap H \times N)$ for some $H \subseteq G$, $N \subseteq M$. The other is to leave G and M unaltered and to consider the context (G, M, J) for some **subrelation** $J \subseteq I$. In both cases, additional conditions are necessary to obtain an interesting construction.

A subrelation $J \subseteq I$ is **closed** if every concept of (G, M, J) is also a concept of (G, M, I). It is easy to check whether a given subrelation $J \subseteq I$ is closed: A necessary and sufficient condition is that $\{g\}^J$ $(:= \{m \in M \mid (g, m) \in J\})$ is an intent and $\{m\}^J$ is an extent of (G, M, I) for each object $g \in G$ and each attribute $m \in M$. Closed subrelations correspond to complete sublattices:

Theorem 3 *S is a complete sublattice of $\underline{\mathfrak{B}}(G, M, I)$ if and only if $S = \underline{\mathfrak{B}}(G, M, J)$ for some closed subrelation $J \subseteq I$.*

A subcontext $(H, N, I \cap H \times N)$ is called **compatible** if $(A \cap H, B \cap N)$ is a concept of $(H, N, I \cap H \times N)$ whenever (A, B) is a concept of (G, M, I). For doubly founded contexts there is a simple way to recognize compatible subcontexts: They are precisely the **arrow-closed** subcontexts, i.e. those with

$$h \in H, h \nearrow m \Rightarrow m \in N \quad \text{and} \quad n \in N, g \swarrow n \Rightarrow g \in H.$$

After reducing the formal context (as can always be done in the doubly founded case) there is an even simpler description using a transitive closure of the arrows: Define a binary relation $\swarrow\!\!\!\swarrow \subseteq G \times M$ to be the smallest relation satisfying

1. If $g \swarrow m$ then $g \swarrow\!\!\!\swarrow m$,
2. If $g \swarrow\!\!\!\swarrow m$, $h \nearrow m$ and $h \swarrow n$, then $g \swarrow\!\!\!\swarrow n$,

and let $\not\swarrow\!\!\!\swarrow$ denote the complement of this relation.

Theorem 4 *A subcontext $(H, N, I \cap H \times N)$ of a standard context (G, M, I) is compatible if and only if $(G \setminus H, N)$ is a concept of $(G, M, \not\swarrow\!\!\!\swarrow)$.*

As the next theorem shows, compatible subcontexts correspond to factor lattices and thereby to complete congruences of $\underline{\mathfrak{B}}(G, M, I)$. In general, this correspondence needs to be neither one-to-one nor onto. The situation is simpler for doubly founded contexts and in particular for reduced ones.

Theorem 5 *A subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is compatible if and only if*

$$(A, B) \stackrel{\Pi_{H,N}}{\longrightarrow} (A \cap H, B \cap N)$$

defines a complete homomorphism from $\underline{\mathfrak{B}}(G, M, I)$ onto $\underline{\mathfrak{B}}(H, N, I \cap H \times N)$.

Theorem 6 *If (G, M, I) is the standard context of a doubly founded complete lattice $\underline{L}$, then to every complete congruence Θ of $\underline{\mathfrak{B}}(G, M, I)$ there is a unique subcontext $(H, N, I \cap H \times N)$ such that*

$$\Theta = \ker \Pi_{H,N}.$$

A combination of these results yields

Theorem 7 *Let (G, M, I) be doubly founded and reduced. The lattice of complete congruences of $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to $\underline{\mathfrak{B}}(G, M, \not\Swarrow)$.*

This makes it very easy to compute the congruence lattice of any finite lattice $\underline{L}$. The concepts of $\underline{\mathfrak{B}}(J(\underline{L}), M(\underline{L}), \not\Swarrow)$, which can be generated with the above algorithm, are in bijective correspondence with the (complete) congruences, and the factor lattice corresponding to a concept (A, B) of $\underline{\mathfrak{B}}(J(\underline{L}), M(\underline{L}), \not\Swarrow)$ is isomorphic to $\underline{\mathfrak{B}}(G \setminus A, B, I \cap (G \setminus A) \times B)$.

An alternative characterization of congruences is obtained using the fact that certain superrelations of the incidence relation I, the so-called *block relations*, are in 1-1 correspondence with the complete tolerance relations of the concept lattice.

4 Subdirect products and tensor products

Closed subrelations and compatible subcontexts can be combined neatly to describe subdirect products. Note that we only consider *complete* subdirect products, defined as complete sublattices of the direct product for which the projections are surjective. The first result shows that decompositions can be read off from the formal context:

Theorem 8 *The subdirect decompositions of a doubly founded complete lattice $\underline{L}$ correspond (bijectively) to the families of arrow-closed subcontexts $(G_t, M_t, I \cap G_t \times M_t)$ of its standard context (G, M, I) satisfying $\bigcup G_t = G$ and $\bigcup M_t = M$.*

As a corollary, we obtain

Theorem 9 *A doubly founded complete lattice is subdirectly irreducible if and only if its standard context contains an attribute m with $g \Swarrow m$ for all objects g.*

These characterizations allow us to transfer a well known result to complete lattices:

Theorem 10 *Every doubly founded complete lattice has a subdirect decomposition into subdirectly irreducible factors.*

It is also worthwhile to consider the complete subdirect product as a *construction* method. We first mention that a direct product of complete lattices translates to a sum of formal contexts, defined as follows. Let T be some index set and let (G_t, M_t, I_t), $t \in T$, be a family of formal contexts. For purposes of simplicity assume $G_s \cap G_t = \emptyset = M_s \cap M_t$ for all $s \neq t$. We define the **sum** of these contexts to be

$$\sum_{t\in T}(G_t, M_t, I_t) := \left(\bigcup_{t\in T} G_t, \bigcup_{t\in T} M_t, \bigcup_{t\in T} I_t \cup \bigcup_{s\neq t} G_s \times M_t \right).$$

Theorem 11

$$\underline{\mathfrak{B}}\left(\sum_{t\in T}(G_t, M_t, T_t)\right) \cong \prod_{t\in T} \underline{\mathfrak{B}}(G_t, M_t, I_t).$$

The isomorphism is canonical. Therefore, to every complete sublattice of the direct product, there corresponds a closed subrelation of the sum context. These can be characterized:

Theorem 12 *The complete subdirect products of concept lattices $\underline{\mathfrak{B}}(G_t, M_t, I_t)$ correspond bijectively to those closed subrelations J of the sum context $\sum_{t\in T}(G_t, M_t, I_t)$ that satisfy $J \cap G_t \times M_t = I_t$ for all $t \in T$.*

There is an even more operational characterization, for which we refer to [2]. It uses the notion of bonds between formal contexts: A **bond** from a context (G_1, M_1, I_1) to (G_2, M_2, I_2) is a relation $J \subseteq G_1 \times M_2$ such that

1. for each $g \in G_1$, the set g^J is an intent of (G_2, M_2, I_2), and
2. for each $m \in M_2$, m^J is an extent of (G_1, M_1, I_1).

In order to turn the subdirect product construction into an operation with a unique result, the notion of a P-lattice is introduced: Let P be some set and define a complete P**-lattice** to be a pair $(\underline{L}, \alpha)$, where $\underline{L}$ is a complete lattice and α is a function that maps P onto a generating set in $\underline{L}$. For a family $(\underline{L}_t, \alpha_t)$, $t \in T$, of P-lattices the P**-product** is defined to be the complete sublattice of the direct product which is generated by $\{(\alpha_t(p) \mid t \in T) \mid p \in P\}$. If each $\underline{L}_t$ is represented as a concept lattice $\underline{\mathfrak{B}}(G_t, M_t, I_t)$, then each $\alpha_t(p)$ is some concept, say, $\alpha_t(p) =: (A_t, B_t)$.

Theorem 13 *The closed subrelation J of the sum context $\sum_{t\in T} \underline{\mathfrak{B}}(G_t, M_t, I_t)$ which corresponds to a P-product of P-concept lattices $\underline{\mathfrak{B}}(G_t, M_t, I_t)$ is characterized by the following properties:*

1. *$J \cap G_t \times M_t = I_t$ for all $t \in T$,*

2. *For $s, t \in T, s \neq t$, $J \cap G_s \times M_t$ is the smallest bond from (G_s, M_s, I_s) to (G_t, M_t, I_t) containing the sets $A_s^p \times B_t^p$ for all $p \in P$.*

There is also a natural definition of a direct product of formal contexts. The corresponding lattice construction is the tensor product of complete lattices. The definitions are simple:

The **direct product** of formal contexts (G_t, M_t, I_t), $t \in T$, is given by

$$\bigtimes_{t \in T} (G_t, M_t, I_t) := \left(\bigtimes_{t \in T} G_t, \bigtimes_{t \in T} M_t, \nabla \right),$$

where for $g := (g_t \mid t \in T)$ and $m := (m_t \mid t \in T)$ we have

$$(g, m) \in \nabla :\Longleftrightarrow (g_t, m_t) \in I_t \quad \text{for some } t \in T.$$

The **tensor product** of complete lattices $\underline{L_t}$, $t \in T$, is defined as

$$\bigotimes_{t \in T} \underline{L_t} := \underline{\mathfrak{B}} \left(\bigtimes_{t \in T} (L_t, L_t, \leq) \right).$$

For a complete lattice $\underline{L}$ there are usually many formal contexts with $\underline{L} \cong \underline{\mathfrak{B}}(G, M, I)$. It was mentioned in the basic theorem that $(L, L, \leq)$ is one of them. But it is not crucial that this particular context was used in the definition of the tensor product, as the next theorem shows.

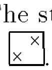
Theorem 14

$$\bigotimes_{t \in T} \underline{\mathfrak{B}}(G_t, M_t, I_t) \cong \underline{\mathfrak{B}} \left(\bigtimes_{t \in T} (G_t, M_t, I_t) \right).$$

The arrow relations of a product context are given by those of the factors. This yields

Theorem 15 *The congruence lattice of a tensor product of finitely many doubly founded complete lattices is isomorphic to the tensor product of the congruence lattices of the factors:*

$$\mathfrak{C} \left(\bigotimes_{i=1}^{n} \underline{L_i} \right) \cong \bigotimes_{i=1}^{n} \mathfrak{C}(\underline{L_i}).$$

For example, the congruence lattice of the free bounded distributive lattice FBD(n) with n generators is the boolean lattice with 2^n atoms, since FBD(n) is isomorphic with the n-th tensor power of the three element chain FBD(1). The standard contexts of $\underline{D_2} \times \underline{D_2}$ (which is the congruence lattice of FBD(1)) is . Its n-th power is isomorphic to $(S, S, \neq)$, where $|S| = 2^n$, and $\underline{\mathfrak{B}}(S, S, \neq) \cong (\mathfrak{P}(S), \subseteq)$.

In the distributive case our definition of a tensor product coincides with those of other authors:

Theorem 16 *The tensor product of completely distributive complete lattices is completely distributive.*

Many further properties of the tensor product can be found in [2]. The fact that the tensor product is distributive over the P-product can be utilized for constructions, e.g. for subdirect representations of the free distributive lattices. Its is possible to define a subdirect product of formal contexts and thereby to obtain a lattice construction of the **subtensorial product**.

5 Lattice properties

A concept lattice is uniquely determined by its formal context. Thus, in principle, every structural property can be read off from the incidence relation. This is of importance e.g. for algorithms: a lattice feature that can easily be detected in a formal context, is "algorithmically harmless". Fortunately this is the case for many of the commonly used properties, at least for doubly founded lattices. Most of the results which we formulate below for the finite case generalize to doubly founded lattices, see [2].

The arrow relations $\nearrow$ and $\swarrow$ and their "transitive closure" $\swarrow\!\!\!\swarrow$ are easy to compute. Once this is done, the following theorem can be applied without much effort:

Theorem 17 *A finite lattice with standard context* (G, M, I) *is*

distributive *if and only if* $\quad g \swarrow\!\!\!\!\nearrow m, g \nearrow n \;\Rightarrow\; m = n,$

join distributive[1] *if and only if* $\quad g \nearrow m, h \nearrow n \;\Rightarrow\; g = h,$

join semidistributive[2] *if and only if* $\quad g \swarrow\!\!\!\!\nearrow m, h \swarrow\!\!\!\!\nearrow m \;\Rightarrow\; g = h,$

semimodular *if and only if* $\quad g \swarrow m, g \swarrow n, (h, m) \in I, (h, n) \notin I$ *implies*

$$\exists_{p \in M} (h, p) \notin I, (g, p) \in I, m' \cap n' \subseteq p',$$

simple *if and only if* $\quad g \swarrow\!\!\!\swarrow m$ *for all* $g \in G$ *and all* $m \in M,$

subdirectly irreducible *if and only if*

$$g \swarrow\!\!\!\swarrow m \text{ for all } g \in G \text{ and some } m \in M.$$

More precise information on the complexity of these and similar properties are compiled in Skorsky's thesis [3].

[1] i.e. locally distributive

[2] i.e. satisfies $\mathrm{SD}_\vee$

Determining the order dimension of an ordered set is known to be a $\mathcal{NP}$-complete problem. Since the dimension of $(P, \leq)$ is equal to that of its MacNeille-completion $\underline{\mathfrak{B}}(P, P, \leq)$, it cannot be an easy task to compute the dimension of a lattice from its context. But there is at least a nice reformulation that sometimes is quite illuminative. It uses the notion of a **Ferrers relation**, i.e. a relation $F \subseteq G \times M$ satisfying

$$(g, m) \in F, (h, n) \in F, (h, m) \notin F \;\Rightarrow\; (g, n) \in F.$$

The **Ferrers dimension** of a formal context (G, M, I) is defined as the smallest number of Ferrers relations whose intersection is I. Since the complement of a Ferrers relation is again a Ferrers relation, the Ferrers dimension is also the smallest number of Ferrers relations the union of which is $G \times M \setminus I$. The latter is more convenient for practical work, because the Ferrers relations can be entered into the "empty cells" of the formal context.

Theorem 18 *The order dimension of a concept lattice $\underline{\mathfrak{B}}(G, M, I)$ equals the Ferrers dimension of (G, M, I).*

A relation $F \subseteq G \times M$ is Ferrers if and only if $\underline{\mathfrak{B}}(G, M, F)$ is a chain. Using the length of these chains as parameters, one can obtain an analogous theorem for the **k-dimension** of finite lattices.

References

[1] A list of publications can be found under

http://www.mathematik.th-darmstadt.de/ags/ag1.

[2] Bernhard Ganter and Rudolf Wille. *Formale Begriffsanalyse – Mathematische Grundlagen.* Springer Verlag 1996.

[3] Martin Skorsky. *Endliche Verbände - Diagramme und Eigenschaften.* Ph.D. thesis, TH Darmstadt 1992. Shaker Publications.

[4] Frank Vogt. *Formale Begriffsanalyse mit C^{++}: Datenstrukturen und Algorithmen.* Springer Verlag 1996.

[5] Rudolf Wille. Restructuring lattice theory: an approach based on hierarchies of concepts. In: Ivan Rival, editor, *Ordered sets*, Reidel, Dordrecht–Boston (1982), 445–470.

[6] Rudolf Wille. *Introduction to Formal Concept Analysis.*

[7] Rudolf Wille. *Conceptual landscapes of knowledge: a pragmatic paradigm for knowledge processing.*